Indirect Measurements Via Polynomial Chaos Observer

A. Smith¹, A. Monti¹, F. Ponci¹
¹Department of Electrical Engineering, University of South Carolina Columbia, SC (USA)

Abstract – This paper proposes an innovative approach to the design of algorithms for indirect measurements based on a Polynomial Chaos Observer (PCO). A PCO allows the introduction and management of uncertainty in the process. The structure of this algorithm is based on the standard closed-loop structure of an observer originally introduced by Luenberger. This structure is here extended to include uncertainty in the measurement and in the model parameters in a formal way. Possible applications of this structure are then also discussed.

Keywords – Kalman filtering, observability, observers, stochastic system, uncertainty, state-space method.

I. INTRODUCTION

The need of indirect measurement is typical of many control applications. The increasing success of control structures based on the complete measurement of the state are a typical reference case. It is usually not feasible in an economic and sometime also in a technical sense to include a sensor for each state variable. For this reason, using the concept of state observability, it is common practice to perform the operation of state measurement starting from a limited set of measurements. Such a process increases the issue of uncertainty. Not only the measurement itself will be affected by uncertainty but also the model of the system will be affected by uncertainty in the parameters and in the structure.

This paper proposes a new observer structure able to reconstruct the state under uncertainty conditions thanks to a stochastic model based on Polynomial Chaos Theory. This approach represents an evolution with respect to classical solutions such as Extended Kalman Filter where parameter uncertainty is managed as a noise problem.

In this paper the authors introduces Polynomial Chaos Theory and then focuses on the mathematical formalization of the PCO. Also, the methodologies for the design of this observer are presented as well as a complete example of application.

II. POLYNOMIAL CHAOS THEORY

Polynomial Chaos is a technique that uses a polynomial based stochastic space to represent and propagate uncertainty in the form of PDFs. This concept was first introduced by N. Wiener in 1938 as “Homogeneous Chaos” [1]. The theory evolved into the Wiener-Askey Polynomial Chaos with the extension of theory to the entire Askey scheme of orthogonal polynomials [2]. Polynomial Chaos can be described through the relationship and the description of a generic second order random process in (1)

\[ X(\theta) = a_i \Psi_i(\xi_i) + \sum_{i=1}^{n_p} a_i \sum_{j=1}^{n_p} \Psi_i(\xi_j) \Psi_j(\xi_i) + \sum_{i=1}^{n_p} \sum_{j=1}^{n_p} \Psi_i(\xi_j) \Psi_j(\xi_i) \]

(1)

Where:
- \( X \) is the random process under analysis
- \( a_i \) are the coefficients of the expansion
- \( \Psi_i \) the polynomials of the selected base
- \( \xi_i \) a random variable with suitable PDF defined according to the polynomial base

The second order random process from (1) represents the infinite space of a complete multivariable orthogonal polynomial space.

The shape of the represented PDF is created in this stochastic space. The completeness of the space allows for the accurate representation of any PDF using any polynomial base. Certain polynomial bases can be chosen to represent given PDF with the fewest number of terms. Considering the properties of the polynomial base and the definition of the variable \( \xi_i \), a perfect Gaussian distribution can be described using only two terms of the Hermite polynomial base, while it takes more terms to represent the Gaussian distribution with the Legendre polynomials. The Legendre polynomial can however represent the Uniform distribution with only two terms.

For practical applications, the stochastic space described by (1) must be limited to a finite number of dimensions. The selection of this number is based on the number of independent source of uncertainty (i.e. the number of independent variables \( \xi_i \) used to describe the process) in the system \( n_i \). The maximum order for the polynomial base has also to be defined \( (n_p) \). Given the two values \( n_i \) and \( n_p \), the total number of terms needed for the description of each variable in the system is defined by

\[ P = \left( \frac{(n_i + n_p)!}{n_i! n_p!} \right) - 1 \]

(2)

The spectral representation of a PDF on this limited space is in (3).
\[ Y(t) = \sum_{i=0}^{P} y_i(t)\Psi_i \]  

The variable \( y_i \) represents the \( i \)th coefficient of \( i \)th term of the polynomial basis while the symbol \( \Psi_i \) actually is the \( i \)th term of the polynomial basis. The size of the multivariable polynomial basis is based partly on the number of uncertainties in the system. Polynomial math can therefore be used to form mathematical relationships between the various PDF represented on the polynomial space. This allows the resultant PDF to be calculated within the polynomial space.

Polynomial Chaos has been applied to numerous different fields of study including fluid dynamics and circuit simulation \([3][4][5][6]\). A previous application to the field of study is reported in \([7]\).

### III. GENERIC PCT MODEL

Let us start considering a linear, time-invariant, dynamic system in state-space form:

\[ \dot{x}(t) = Ax(t) + Bu(t) \]
\[ y(t) = Cx(t) \]  

where:  
\( x(t) \in \mathbb{R}^n \) is the system state vector  
\( u(t) \in \mathbb{R}^p \) is the input vector,  
\( y(t) \in \mathbb{R}^m \) is the output vector and  
\( A, B, C \) are the system matrices with appropriate dimensions.

Expanding a system through PCT employs orthogonal polynomials in the Askey scheme to manage uncertain variable/variables in the governing equation. This expansion technique associates individual random events with independent random variables, \( \xi_i \).

In the model being expanded through PCT it is assumed that the probability density function of the \( n_v \) uncertain variables are known. These variables or parameters can be described, as in (3), as stochastic uncertain parameters with a PCT base of \( n_p \) order. A new set of deterministic state equations can then be found by projecting the stochastic uncertain equation onto the random space spanned by the orthogonal polynomial basis, \( \Psi_i \), and by taking the inner product of the equation of each basis. The inner product can be determined by taking the Galerkin projection, where the Galerkin projection is realized by the integration of each component of the system with the polynomial basis.

\[ \langle \Psi_i, \Psi_j \rangle = \frac{1}{\Omega} \int_{\Omega} \Psi_i \Psi_j w(\xi) \, d\xi \]

where:
\( w(\xi) \) depends on the choice of basis in the Askey scheme  
\( \Omega \) is the region for which the chosen basis is valid.

The solution of this new set of state equation provides the coefficients of the polynomial chaos expansion of the state variables. The order zero of such expansion represents the most likely value. This expanded state-space PCT model can be described as:

\[
\dot{x}_{pct}(t) = A_{pct}x_{pct}(t) + B_{pct}u_{pct}(t) \\
y_{pct}(t) = C_{pct}x_{pct}(t)
\]  

where:
\( x_{pct}(t) \in \mathbb{R}^{n_{pct}} \) is the system state vector  
\( u_{pct}(t) \in \mathbb{R}^{n_{u_{pct}}} \) is the input vector,  
\( y_{pct}(t) \in \mathbb{R}^{n_{y_{pct}}} \) is the output vector and  
\( A_{pct}, B_{pct}, C_{pct} \) are the system matrices with appropriate dimensions.

The states of PCT model is observable from the output of the original system if

\[
W_o = \begin{bmatrix} C_{pct} \\ C_{pct}A_{pct} \\ \vdots \\ C_{pct}A_{pct}^{n_{y_{pct}}-1} \end{bmatrix}
\]  

has a rank equal to \( n_{pct} \)

where \( C_{pct} \) is the matrix structure such that output is the measured, (expected), states of the original system.

If the system is observable a close loop state observer can be designed. Let us assume the set of variables \( y \) defines the measurable variables, the observer could be defined as:

\[
\dot{\hat{x}}_{pct}(t) = A_{pct}\hat{x}_{pct}(t) + B_{pct}u_{pct} + K_{pct}(y_{pct}(t) - C_{pct}\hat{x}_{pct}(t))
\]

where:  
\( \hat{x}_{pct}(t) \in \mathbb{R}^{n_{y_{pct}}} \) is the estimation of the state system vector.

The gain \( K_{pct} \) is a design parameter and different strategies for obtaining the gain matrix can be introduced. Such strategies include pole-placement, optimal and sub-optimal Kalman filter design. In the pole-placement design \( K_{pct} \) is chosen such that the poles of the observer are at desired locations. Choosing poles on the left hand plane ensures that the PCO is asymptotically stables and the error will always converge to zero \([8]\). \( K_{pct} \) can also be found in terms of a gain that minimizes the mean square of the error as in the Kalman filter design. The Kalman filter design is an optimal strategy that would require \( K_{pct} \) to be time-varying. The time-varying gain in the optimal Kalman filter design reaches a steady-state value far from final time. Thus a steady-state or sub-optimal gain can be found which relax the requirement of \( K_{pct} \) to be time-varying. The optimal and sub-optimal Kalman \( K_{pct} \) can be found using the Riccati, steady-state Riccati equation respectively \([8]\).

The formulation of the model detailed in (8) is able to include a description of the uncertainty propagation created by the model uncertainty itself.
A more comprehensive uncertainty propagation process can be defined if we assume, as it is reasonable to do, that the measurement itself is affected by uncertainty. This uncertainty will introduce a new dimension to the PCT space. Instead of an uncertain parameter we have now an uncertain input. In any case the order of complexity will be increased.

IV. EXAMPLE OF PCT EXPANDED SYSTEM WITH UNCERTAINTY ON A MEASURED PARAMETER

Let us consider a linear time invariant model such as the average model of a buck DC/DC converter (see Figure 1)

![Buck converter schematic](image)

Figure 1: Buck converter schematic

Assuming for sake of simplicity that the resistances associated with the inductance and capacitor are null, so to have an ideal lossless converter.

The averaged state-space model of the converter is given by:

\[
\begin{align*}
\frac{d}{dt}[i(t)] &= \left[ \begin{array}{ccc}
0 & -\frac{1}{L} & 0 \\
\frac{1}{C} & -\frac{1}{CR} & 0 \\
\end{array} \right] \left[ \begin{array}{c}
i(t) \\
v(t) \\
\end{array} \right] + \left[ \begin{array}{c}
\frac{V_{cc}}{L} \\
0 \\
\end{array} \right] dt
\end{align*}
\]

where
- \( L \) is the value of the inductor
- \( C \) is the value of the capacitor
- \( R \) is the value of the load resistor
- \( i(t) \) is the current through the inductor
- \( v(t) \) is the voltage across the capacitor

The PCT expansion of the equation (9) with uncertainty on \( R \) using the Hermite polynomial basis in state space formulation is given by

\[
\begin{align*}
\frac{di_0(t)}{dt} &= \left[ \begin{array}{ccc}
0 & 0 & \frac{1}{L} \\
0 & 0 & -\frac{1}{L} \\
\frac{1}{C} & 0 & -G_0 \\
\frac{1}{C} & 0 & -G_1 \\
0 & \frac{1}{C} & -G_0 \\
0 & \frac{1}{C} & -G_1 \\
\end{array} \right] \left[ \begin{array}{c}
i_0 \\
i_1 \\
v_0 \\
v_1 \\
\end{array} \right] + \left[ \begin{array}{c}
\frac{V_{cc}}{L} \\
0 \\
0 \\
0 \\
\end{array} \right] dt
\end{align*}
\]

where:
- \( i_0 \) is the expected value of the inductor current
- \( i_1 \) is the uncertainty on the expected value of the inductor current
- \( v_0 \) is the expected value of the capacitor voltage
- \( v_1 \) is the uncertainty on the expected value of the capacitor voltage

From equation (10)

\[
A_{pct} = \begin{bmatrix}
0 & 0 & -\frac{1}{L} & 0 \\
0 & 0 & 0 & -\frac{1}{L} \\
1 & 0 & -G_0 & -G_1 \\
0 & 1 & -G_0 & -G_1 \\
\end{bmatrix}
\]

and

\[
C_{pct} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
\end{bmatrix}
\]

Constructing the observability matrix \( W_o \), it can be shown that

\[
\text{rank}(W_0) = 4 \quad \forall G_i
\]

Therefore, assuming that the outputs of the measured system is the most-likely value of the measured variable, the uncertain states of the PCT model are observable from the measured system for all values of \( G_i \).

Now suppose the measurement introduces a new element of uncertainty, it can be shown that the total order of the state space formulation of the observer grows to 8 while the matrix \( C_{pct} \) will be modified as in the following:

\[
C_{pct} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\end{bmatrix}
\]

V. SIMULATION RESULTS

The following parameters for the average model of a buck converter were used to obtain the results on the uncertainty on the outputs given an uncertainty of the load conductance \( G_0 \).

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( V_{cc} )</td>
<td>1V</td>
</tr>
<tr>
<td>( L )</td>
<td>1 mH</td>
</tr>
<tr>
<td>( C )</td>
<td>100( \mu )F</td>
</tr>
<tr>
<td>( G_0 )</td>
<td>0.3( \Omega )</td>
</tr>
<tr>
<td>( G_1 )</td>
<td>0.1( \Omega )</td>
</tr>
</tbody>
</table>
Using the parameters in Table 1 the behavior of the system under uncertain conditions was studied. A step variation is given on the input channel that is assumed to be the duty cycle of the power electronic switch. The operations of five different models are going to be compared:

1. **The plant model.** This is supposed to be the real system under analysis. All the parameters, except the load are equal to the rated value and the load is varied within the range of uncertainty and outside the range of uncertainty.

2. **An open loop uncertain model estimator.** This is a PCT model that receives as input the duty cycle and calculates the state and its uncertainty.

3. **A closed loop PCT Observer (Luenberger).** The loop gain of the observer has been calculated to have poles faster than the system to estimate the states and its uncertainty.

4. **An Optimal Kalman PCT filter.** The loop gain of the observer is time-varying and it minimizes the mean square of the error, to estimate the states and its uncertainty.

5. **A Steady-state Kalman PCT filter.** The loop gain of the observer is time-invariant and it sub-optimally minimizes the mean square of the error, to estimate the states and its uncertainty.

Figure 2 shows a comparison between the plant model and the PCT Open loop state estimator along with the PCT optimal Kalman filter, PCT Steady-State Kalman filter and the PCT Luenberger observer. The trace in the middle is the plant transient while the two external traces represent the bounds of the uncertainty mode. Coherently with the theory, being the variation of the parameter within the range the a priori assumed range, the transient evolution is within the boundary.

For the PCT open-loop/observer/filter it is interesting to analyze not only the most likely value but also the estimation of the boundary limit for the current. One such analysis is perturbing the load resistor in and out of the boundary of the described uncertainty. In figure 3 and 4 the load was changed such that it was within the uncertainty boundary, then above, back within uncertainty boundary and then below the uncertainty boundary. From the figure 3 and 4 it can be seen that the Luenberger and Kalman estimators can be designed to track the measured values or to provide the bounds of the measured value given the uncertainty. It can also be seen that the open loop estimator can only be used to give the bounds. This is results is expected since an open loop estimator is not affected by changes in system.

Now let us assume that we do not have a current measurement available but only the voltage across the capacitor. In this case we really implement a process of indirect measurement. The same simulation has been repeated simply changing this assumption on the number of sensors.

It is interesting to see that, in Figure 4, in this new situation the observer is not showing superior performance in comparison with the simple open loop estimator justification for this is that the estimation of the capacitor voltage performed by the PCT observer is reasonably correct. As a result of that the observer feedbacks a null error not improving the performance of the closed loop operation.

This behavior can be physically justified. The steady state value of the voltage under lossless conditions does not depend on the load. As a consequence the uncertain parameter is not affecting the measured variable and then the error can not be propagated in the model. A more realistic model including loss in the passive component should be sufficient to remove the problem. In any case it is interesting to observe that in the process of indirect measurement the selection of the observer variable is critical for the overall performance of the system. The PCT model is actually able to predict if the sensor is going to be effective for the observer structure.
Figure 3: shows that PCT observer/filter can be designed to track the state changes when a change occurs in the uncertain parameter. It also shows that the PCT open-loop estimator can be used to determine when the uncertain parameter is within specified uncertainty.

Figure 4: shows that PCT observer/filter can be designed determine when the uncertain parameter is within specified uncertainty.

Figure 5: Comparison among plant voltage and most likely value of the PCT state estimators and their bounds.

VI. CONCLUSIONS

This paper introduced an original structure for indirect measurements under uncertainty. The theory of the approach has been tested with reference to a second order model of a DC/DC power converter.

This paper demonstrated that a closed-loop estimator can be designed to estimate the uncertainty on output states and in-order to perform the estimation a model of the system must be first expanded using PCT. Structuring the PCT model of the system in state-space format allows for the design an estimator through pole-placement or through an optimal technique such as the Kalman filter.

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REFERENCES


