Uncertainty and Worst-Case Analysis in Electrical Measurements Using Polynomial Chaos Theory
Anton H. C. Smith, Antonello Monti, Senior Member, IEEE, and Ferdinanda Ponci

Abstract—In this paper, the authors propose an analytical method for estimating the possible worst-case measurement due to the propagation of uncertainty. This analytical method uses polynomial chaos theory (PCT) to formally include the effects of uncertainty as it propagates through an indirect measurement. The main assumption is that an analytical model of the measurement process is available. To demonstrate the use of PCT to assess a worst-case measurement, the authors present two examples. The first one involves the use of PCT to estimate the possible worst case of a measurement due to the propagation of parametric uncertainty of a low-pass filter. This case study concerns the analysis of nonlinear effects on the propagation of uncertainty of a signal-conditioning stage used in power measurement. In this paper, the PCT method is applied to determine the probability density function (pdf) of magnitude and phase of the frequency response of the filter and their impact on the power measurement. Of particular interest is the use of PCT to determine the worst-case, expected-case, and best-case effects of the filter, avoiding the reconstruction of the complete pdf of the filter output. The results illustrate the potential of this method to determine the significant boundary of measurement uncertainty, even when the uncertainty propagates through a nonlinear nonpolynomial function. In the second example, the authors use PCT to perform a worst-case analysis for an indirect measurement of a loop impedance. For both examples, the PCT method is compared with the numerical Monte Carlo analysis and the analytical method described in the guide on uncertainty on measurements (GUM).

Index Terms—Electric variables measurement, impedance measurement, polynomials, power measurement, uncertainty.

I. INTRODUCTION

The estimation of uncertainty is a critical step in any measurement process. In some measurement processes, a direct measurement of the desired measurand is not possible. However, through the use of a model, an indirect evaluation of the measurand can be obtained. Therefore, the estimation of the individual sources of uncertainty is an important part of the quantification of the uncertainty, but it must be integrated with an analysis of the propagation of the uncertainty through the model. A typical source of measurement uncertainty is the parametric uncertainty in the test circuits or in the electronics of the signal-conditioning section used to make the measurement.

This has led to the choice of the two examples. The examples are intended as an illustration of the proposed method and, thus, refer to simplified circuits whose elaborations are immediately understandable.

This paper presents a novel approach to using polynomial chaos theory (PCT) expansion to find the worst case in uncertainty propagation. The analysis is performed with reference to two examples: 1) a simple second-order analog filter [1] and 2) an indirect method for impedance measurement in power systems [2]. We focus our work on the estimation of the most likely case and the two extreme cases defining the tails of the probability density function (pdf). The tails of the pdf in many applications coincide with the worst cases; however, occasionally, one of the tails of the pdf can actually be considered as the best case. For example, if the filter under analysis is part of a closed-loop control system, one of the tails is the minimum phase shift in the frequency domain (best case), while the other is the maximum phase shift (worst case). Purely, from a measurement perspective, both cases can be considered as worst case in the sense that they represent the maximum distance from the most likely value.

This paper is structured as follows. Section II presents a brief introduction of PCT, while Section III contains a discussion of the theory of worst-case analysis in the PCT domain. Section IV provides the PCT modeling of an uncertain second-order Sallen–Key filter. In Section V, the propagation of the uncertainty of the filter to a power measurement is quantified, and a demonstration is presented on how to use PCT to obtain the worst-case measurement without reconstructing the pdf. Section VI compares the PCT results from the Sallen–Key filter with that of using Monte Carlo analysis and with the method discussed in the guide on uncertainty on measurements (GUM) and in [3]. Section VII demonstrates the uncertainty estimation of an indirect impedance measurement using PCT. Furthermore, in this section, a comparison is made with the method discussed in the GUM and in [2]. Final conclusions are reported in Section VIII.

II. PCT

In 1938, Wiener introduced homogeneous chaos expansion in [4], where he discussed the use of Hermite polynomials and homogeneous chaos. Wiener used the Hermite polynomials in a stochastic space to represent and propagate uncertainty in the form of a pdf. This scheme was later expanded to include the whole Askey scheme of orthogonal polynomials and was renamed Wiener–Askey polynomial chaos [5]. PCT is a spectral expansion of random variables that approximates a random process or, in general, a random variable, by means of a
complete and orthogonal polynomial basis of random variables. Using PCT, the spectral expansion of a second-order random variable can be described as

\[ g(\xi_1, \ldots, \xi_i) = \sum_{n=0}^{\infty} x_n \Phi_n(\xi_1, \ldots, \xi_i) \]  

(1)

where

- \( g(\xi) \) random variable or function under analysis;
- \( x_n \) coefficients of the expansion;
- \( \Phi_n \) polynomials of the selected base;
- \( \xi_i \) random variables with a pdf defined according to the polynomial base (selection of the convenient base minimizes the computational effort).

The spectral expansion is an infinite series, but for practical purposes, it must be limited to a finite number \( P \) of terms. That is,

\[ g(\xi_1, \ldots, \xi_i) = \sum_{n=0}^{P} x_n \Phi_n(\xi_1, \ldots, \xi_i). \]  

(2)

The value of \( P \) of the truncated PCT expansion can be found, given two values: 1) the number of independent sources of uncertainty \( (n_v) \) and 2) the maximum order for the polynomial base \( (n_p) \) (the total number of terms). The value of \( P \) of the truncated PCT expansion is given as

\[ P = \left( \frac{(n_v + n_p)!}{n_v!n_p!} \right) - 1. \]  

(3)

Using a truncated PCT expansion allows the use of the theory for practical applications. Polynomial chaos has been applied to numerous fields of study, including fluid dynamics [6] and circuit simulation [7]. Previous applications involving measurement are reported in [1] and [8].

III. WORST-CASE ANALYSIS

The use of PCT allows the representation of a random variable as a polynomial series expansion [4]–[9]. One of the key aspects of this paper is to present how the analysis of the polynomial function allows a quick estimation of the tails of the pdf of the variable under consideration.

A generic random variable can be expressed through PCT expansion, as in (2). From (2), the extreme cases can be expressed using (4). This calculation requires finding the value(s) of \( \xi \) that evaluates the expanded variable under consideration at a maximum or minimum, i.e.,

\[ \sup_{\Omega} g(\xi) = \sup_{\Omega} \left( \sum_{n=0}^{P} x_n \Phi_n(\xi) \right) \]

\[ \inf_{\Omega} g(\xi) = \inf_{\Omega} \left( \sum_{n=0}^{P} x_n \Phi_n(\xi) \right) \]  

(4)

where \( \Omega \) is the region where \( \Phi_n(\xi) \) is defined.

In many cases, finding \( \sup(\xi) \)/\( \inf(\xi) \) can be solved analytically, given the polynomial nature of the base functions. The complexity of this calculation is mostly related to the number of terms used in the PCT expansion. In particular, given the monotonic characteristics of many base functions, it becomes an easy constrained problem on the frontier of the domain of the variable \( \xi \). Concerning the definition of the frontier \( \Omega \), it is important to clarify that different approaches must be considered, depending on the characteristics of the reference distribution adopted for \( \xi \). If we adopt a Legendre base, \( \xi \) will have a uniform distribution, and then, the frontier is automatically defined; if \textit{vice versa}, we adopt a Hermite expansion, having \( \xi \) as a Gaussian distribution, we define the limit from an engineering standpoint to a reasonable value for \( \xi \) such as three times the standard deviation.

In this paper, two examples are presented to demonstrate the use of this concept where most of the computational effort has been automated by means of Maple scripts. For both examples, the Maple scripts took less than 44 s, including the display of the pdf for comparison.

IV. SALLEN–KEY FILTER EXAMPLE

The low-pass Sallen–Key filter is an active filter whose topology is shown in Fig. 1. This simple second-order filter provides an interesting case where the nonlinear propagation of uncertainty is present.

With reference to Fig. 1, the transfer function of the low-pass Sallen–Key filter is given as

\[ H(s) = \frac{1}{1 + C_2(R_1 + R_2)s + C_1C_2R_1R_2s^2}. \]  

(5)

From the transfer function (5), the frequency response of the filter can be obtained where the magnitude of the frequency response of this filter is

\[ |H(j\omega)| = \sqrt{\left(1 - C_1C_2R_1R_2\omega^2\right)^2 + \left(-C_2R_1\omega - C_2R_2\omega\right)^2} \]  

(6)

where

\[ F(\omega) = 1 + (-2C_1C_2R_1R_2 + C_2^2R_1^2 + 2C_2^2R_1R_2 + C_2^2R_1^2)\omega^2 + C_2^2R_1^2\omega^4 \]  

(7)

and the phase of the frequency response is

\[ \angle H(j\omega) = -\tan^{-1}\left(\frac{-C_2R_1\omega - C_2R_2\omega}{1 - C_1C_2R_1R_2\omega^2}\right). \]  

(8)
Any uncertainty in the electrical parameters results in magnitude and phase uncertainty. Therefore, the analysis in the PCT domain can be split into the following two parts:

1) impact of the parameter uncertainty on the amplitude;
2) impact of the parameter uncertainty on the phase

where the uncertain parameter may be any of the lumped equivalent circuit parameters.

Both the evaluation of the phase and the amplitude involve the use of a nonlinear function, making the analysis of the propagation of the uncertainty nontrivial, particularly if the focus is on the identification of the extreme cases.

Focusing first on the analysis of the phase, the PCT expansion of the phase can be found by considering a function \( f(\omega) \) such that \( \angle H(j\omega) = -\tan^{-1}(f(\omega)) \); in other words

\[
f(\omega) = \frac{-C_2R_1\omega - C_2R_2\omega}{1 - C_1C_2R_1R_2\omega^2}.
\]

(9)

It is more convenient to rewrite (9) as in (10) to properly manage the division operation in the PCT domain. That is,

\[
(1 - C_1C_2R_1R_2\omega^2)f(\omega) = -C_2R_1\omega - C_2R_2\omega.
\]

(10)

Equation (10) consists of functions with uncertain variables on the left-hand side (LHS) and right-hand side (RHS). These LHS and RHS equations can be expanded separately and later equated and solved for \( f(\omega) \) as follows:

\[
\begin{align*}
\text{LHS} &= (1 - C_1C_2R_1R_2\omega^2)f(\omega) \\
\text{RHS} &= -C_2R_1\omega - C_2R_2\omega.
\end{align*}
\]

(11)

The next step in the expansion is the substitution of the uncertain parameters with their polynomial expression in the PCT domain. In this case study, it is assumed that the uncertain parameters are \( C_1, C_2, \) and \( R_1, \) and that they have a uniform pdf; thus, according to [5], the convenient base of choice of the PCT expansion is the Legendre base.

The expansion of these uncertain variables can be expressed as

\[
\begin{align*}
C_1 &= \sum_{i=0}^{P} C_{1i}\Phi_i(\xi_1) \\
C_2 &= \sum_{i=0}^{P} C_{2i}\Phi_i(\xi_2) \\
R_1 &= \sum_{i=0}^{P} R_{1i}\Phi_i(\xi_3).
\end{align*}
\]

(12)

Due to the uncertainty of the parameters, function \( f(\omega) \) is also uncertain. Considering that there are three uncertain parameters, the resulting polynomial base is multidimensional, and the generic variable \( \xi \) is actually a vector with three components \( \xi = [\xi_1, \xi_2, \xi_3] \) [5]. Thus, the series decomposition can be written as

\[
f(\omega, \xi_1, \xi_2, \xi_3) = \sum_{i=0}^{P} f(\omega_i)\Phi_i(\xi_1, \xi_2, \xi_3).
\]

(13)

Substituting (12) and (13) into (11) yeilds

\[
\text{LHS} = \left(1 - \sum_{a=0}^{P} C_{1a}\Phi_a(\xi_1)\right) \left(\sum_{b=0}^{P} C_{2b}\Phi_b(\xi_2)\right) \\
&\times \left(\sum_{c=0}^{P} R_{1c}\Phi_c(\xi_3)\right) R_2\omega^2 \\
&\times \sum_{d=0}^{P} f(\omega_d)\Phi_d(\xi_1, \xi_2, \xi_3)
\]

\[
\text{RHS} = \left(-\sum_{e=0}^{P} C_{2e}\Phi_e(\xi_2)\right) \left(\sum_{f=0}^{P} R_{1f}\Phi_f(\xi_3)\right) \omega \\
&\left(-\sum_{g=0}^{P} C_{2g}\Phi_g(\xi_2)\right) R_2\omega.
\]

(14)

The PCT coefficients or both the LHS and RHS can be found by taking the following Galerkin projection:

\[
\begin{align*}
&\text{LHS}_i = \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} \text{LHS}(w)(\Phi_i(\xi_1, \xi_2, \xi_3))d\xi_1d\xi_2d\xi_3 \\
&\text{RHS}_i = \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} \text{RHS}(w)(\Phi_i(\xi_1, \xi_2, \xi_3))d\xi_1d\xi_2d\xi_3
\end{align*}
\]

(15)

where \( w \) is a weighting function determined by the choice of basis in the Askey scheme and the dimension of the variable \( \Phi(\xi) \) (three in this case).

The final step is the determination of the worst case by equating \( \text{LHS}_i = \text{RHS}_i \) and solving for \( f_i(\omega) \), where \( f_i(\omega) \) is the representation of \( f(\omega) \) in terms of PCT coefficients.

Assuming the following characteristics of the uncertain parameters, both capacitors are supposed to have a uniform distribution in the interval [15 \( \mu \)F, 30 \( \mu \)F]; the resistance \( R_1 \) has a uniform distribution within the interval [90 \( \Omega \), 110 \( \Omega \)], \( R_2 = 100 \Omega \), and the analysis is performed at a specific frequency, namely, \( \omega = 2\pi 60 \text{ rad/s} \).

The coefficients of the PCT expansion of the uncertain parameters, which are computed as in [5], are reported in Table I. Thus, \( f(\omega) \), which is reconstructed from \( f_1(\omega) \), is given as

\[
f(\omega = 2\pi 60, \xi) = -3.496 - 0.6439 \xi_3 - 0.2073 \xi_2 \\
-0.1192 \xi_1 - 0.08496 \xi_3^2 - 0.060903 \xi_2^2 \\
-0.003984 \xi_1^2 - 0.0646 \xi_2 \xi_3 \\
-0.04864 \xi_1 \xi_3 - 0.01380 \xi_1 \xi_2.
\]

(16)

It can be shown that the max and min points of \( f(\omega) \) occur at the boundaries; therefore, the extreme cases occur when \( \xi_1 = \xi_2 = \xi_3 = 1 \) and \( \xi_1 = \xi_2 = \xi_3 = -1 \). Table II shows the values of \( f(\omega = 60) \) at these boundary conditions. Fig. 2 shows the distribution of \( f(\omega = 2\pi 60) \) obtained using Monte Carlo, first-order, and second-order expansion using PCT. In Fig. 2, it can be seen that the results obtained using Monte Carlo closely matched the second-order PCT expansion.
To obtain the PCT expansion of $\angle H(j\omega)$, the angle function

$$\angle H(j\omega) = -\tan^{-1}(f(\omega))$$

is expanded using the Taylor expansion, thus resulting to

$$\angle H(j\omega) = -\tan^{-1}(f(\omega)_0) - \frac{f(\omega) - f(\omega)_0}{1 + f(\omega)_0^2} + \frac{f(\omega)_0 (f(\omega) - f(\omega)_0)^2}{(1 + f(\omega)_0^2)^2} + \cdots$$

where $f(\omega)_0$ is a constant and is the expected value of the function not affected by uncertainty.

The PCT coefficients for $\angle H(j\omega)$ can then be obtained by taking the following Galerkin projection:

$$\angle H(j\omega)_n = \frac{1}{\frac{1}{1} \frac{1}{1} \frac{1}{\Phi_n}(\Phi_n)_n}{\frac{1}{\frac{1}{1} \frac{1}{1} \frac{1}{\Phi_n}(\Phi_n)_n}}.$$  \hspace{1cm} (19)

The uncertain variable in (18) is $f(\omega)$, and its PCT expansion is given in (16). Substituting (16) into (18), we can obtain the system equation for the PCT expansion, which can now be substituted in (19). The distribution of $\angle H(j\omega)$ for the same set of parameters as stated above can be seen in Fig. 3.

The PCT expansion of the phase of the frequency response is reported here for the polynomials up to the second order

$$\tan^{-1}(f(\omega = 2\pi 60, \xi))$$

$$= 1.292 + 0.04808 \xi_3 + 0.01532 \xi_2 + 0.008698 \xi_1$$

$$- 0.001656 \xi_3^2 - 0.0002935 \xi_2^2 - 0.00003605 \xi_1^2$$

$$- 0.0002703 \xi_2 \xi_3 + 0.0007409 \xi_1 \xi_3 + 0.0001182 \xi_1 \xi_2.$$  \hspace{1cm} (20)

It can be shown that the max and min points occur at the boundary; therefore, the extreme cases occur when $\xi_1 = \xi_2 = \xi_3 = 1$, $\xi_1 = \xi_2 = \xi_3 = -1$. Table III shows the extreme cases for $H(j(\omega = 2\pi 60))$.

Now, consider the magnitude function of (7). This equation can be rewritten as

$$|H(j\omega)| F(\omega)$$

$$= \sqrt{(1 - C_1 C_2 R_1 R_2 \omega^2)^2 + (-C_2 R_1 \omega - C_2 R_2 \omega)^2}.$$  \hspace{1cm} (21)

Using (21), the PCT expansion for $|H(j\omega)|_n$ for the magnitude of the frequency response can be obtained. Fig. 4 shows the distribution of $|H(j(\omega = 2\pi 60))|$ obtained using Monte Carlo, first-order, and second-order expansion using PCT. In Fig. 4,
the results obtained using Monte Carlo closely matched both
the first- and second-order PCT expansions, i.e.,
\[ |H(j(\omega = 2\pi 60), \xi)| \]
\[
= 0.6376 - 0.02273 \xi_3 - 0.1053 \xi_2 + 0.01313 \xi_1
+ 0.0001164 \xi_3^2 - 0.01232 \xi_2^2 - 0.001261 \xi_1^2
- 0.0008356 \xi_2 \xi_3 - 0.00170830 \xi_1 \xi_3 - 0.0006978 \xi_1 \xi_2.
\] (22)
Notice that the max and min points do not occur for the same
values of the variables as before. Table IV shows the extreme
cases of \( |H(j(\omega = 2\pi 60))| \).

V. POWER MEASUREMENT
The goal of the analysis is to evaluate the uncertainty of
the real power measurement at 60 Hz, assuming the use of the
same filter stage (for filtering both the current and the voltage
channel) discussed in the previous paragraph. In the following,
we assume, without loss of generality but just for the sake of
simplicity, that the load is purely resistive. In addition, all the
calculations are performed in per unit so that it is assumed to
have a unity voltage and a unity current.

By using all the results of the analysis described in the previ-
ous paragraph, we can estimate the propagation of uncertainty
on the power calculation as a product of the two phasors at
60 Hz applied as input to the two filter channels. The same
calculation could be performed, analyzing the impact of the
filter at other frequencies.

VI. COMPARISON WITH MONTE CARLO ANALYSIS AND
WITH THE COMBINED STANDARD UNCERTAINTY
METHOD FOR THE SALLÉ–KEY FILTER
When the measurand is not directly measured, the GUM
affirms the use of the law of propagation of uncertainty [13].
The law of propagation of uncertainty involves combining the
standard uncertainty of the measurands used in the process of
indirect measurement. The “combined standard uncertainty” is
defined by the GUM [10] and commented in [3] as “an indi-
rect measurement obtained combining primary measurements
according to a model can be evaluated as the positive square
root of the sum of variances and covariances of the primary
measurements, weighted on the rate of dependence of the
indirect measurement.” Following what was reported in [3] and
[11], assuming the measurement of interest, \( y \) can be modeled
as a generic function of a set of \( N \) measurements \( x_i \), i.e.,
\[ y = f(x_1, x_2, \ldots, x_i, \ldots, x_N). \] (26)
Assuming that the central limit theorem holds and that \( f(x) \)
is linear, the standard uncertainty \( u(y) \) can be expressed as in
(27), shown at the bottom of the next page, where \( u(x_i) \) is the
standard uncertainty of the \( x_i \) measurement, and \( r(x_i, x_j) \) is
the degree of correlation between \( x_i \) and \( x_j \), which is given by
\[ r(x_i, x_j) = \frac{u(x_i, x_j)}{u(x_i)u(x_j)}. \] (28)
TABLE IV
BEST CASE, EXPECTED CASE, AND WORST CASE CALCULATED FROM THE PCT EXPANSION OF $|H(j\omega = 2\pi 60)|$

<table>
<thead>
<tr>
<th>Worst-case</th>
<th>Expected-case</th>
<th>Best-case</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\xi_1 = 1, \xi_2 = -1, \xi_3 = -1$</td>
<td>$\xi_1 = \xi_2 = \xi_3 = 0$</td>
<td>$\xi_1 = 1, \xi_2 = -1, \xi_3 = -1$</td>
</tr>
<tr>
<td>0.6003</td>
<td>0.6376</td>
<td>0.6758</td>
</tr>
</tbody>
</table>

Fig. 5. Frequency distribution of the Monte Carlo analysis of the power measurement.

Considering the model of power measurement, the linearity hypothesis of the model is not valid. Therefore, this method can only give approximate results, and the goodness of the approximation must be verified [3].

Thus, it is significant to compare the results of the standard uncertainty method with those of the PCT and Monte Carlo methods that can account for the nonlinearity of the measurement function. The uncertainty of the magnitude of the frequency response of the Sallen–Key filter using the GUM approach is given in (29), shown at the bottom of the page, and the uncertainty of the phase is given in (30), shown at the bottom of the page.

For both the Monte Carlo and PCT methods, the pdf of the uncertain parameters are uniform. Let us assume that the pdf of the amplitude and the phase of the transfer function will also result in a uniform distribution. As a result, we can write

$$I = \bar{I} \pm a_i$$
$$V = \bar{V} \pm a_v$$
$$\theta = \bar{\theta} \pm a_\theta$$  \hspace{1cm} (31)

where $\bar{I}, \bar{V}, \bar{\theta}$ means of the respective parameters; $2a_\%$ 100% confidence interval.

Therefore, these uniform pdfs produce a standard uncertainty given in general as

$$u(x) = \frac{a}{\sqrt{3}}.$$  \hspace{1cm} (32)

The power equation is given by

$$P = |V||I| \cos(\theta_I - \theta_V).$$  \hspace{1cm} (33)

Therefore, the uncertainty of the power measurement can be written as in (34), shown at the bottom of the page.

$$u(y) = \sqrt{\sum_{i=1}^{N} \left( \frac{\partial f}{\partial x_i} \right)^2 u^2(x_i) + 2 \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \left( \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j} \right) r(x_i, x_j) u(x_i) u(x_j)}$$  \hspace{1cm} (27)

$$u(|H(j\omega)|) = \sqrt{\left( \frac{\partial}{\partial C_1} |H(j\omega)| u(C_1) \right)^2 + \left( \frac{\partial}{\partial C_2} |H(j\omega)| u(C_2) \right)^2 + \left( \frac{\partial}{\partial R_1} |H(j\omega)| u(R_1) \right)^2}$$  \hspace{1cm} (29)

$$u(\angle H(j\omega)) = \sqrt{\left( \frac{\partial}{\partial C_1} \angle H(j\omega) u(C_1) \right)^2 + \left( \frac{\partial}{\partial C_2} \angle H(j\omega) u(C_2) \right)^2 + \left( \frac{\partial}{\partial R_1} \angle H(j\omega) u(R_1) \right)^2}$$  \hspace{1cm} (30)

$$u(P) = \sqrt{\left( \frac{\partial P}{\partial |I|} \right)^2 u(|I|)^2 + \left( \frac{\partial P}{\partial |V|} \right)^2 u(|V|)^2 + \left( \frac{\partial P}{\partial \theta_I} \right)^2 u(\theta_I)^2 + \left( \frac{\partial P}{\partial \theta_V} \right)^2 u(\theta_V)^2 + 2 \frac{\partial^2 P}{\partial |I| \partial |V|} r(|I|, \theta_I) u(|I|) u(\theta_I) + 2 \frac{\partial^2 P}{\partial |V| \partial \theta_v} r(|V|, \theta_v) u(|V|) u(\theta_v)}$$  \hspace{1cm} (34)
TABLE V

WORST-CASE VALUES CALCULATED USING PCT, MONTE CARLO, AND THE COMBINED UNCERTAINTY METHOD

<table>
<thead>
<tr>
<th>Parameter</th>
<th>PCT</th>
<th>Monte Carlo</th>
<th>Combined uncertainty (GUM)</th>
</tr>
</thead>
<tbody>
<tr>
<td>|H(j\omega=60)</td>
<td>0.6003</td>
<td>0.6006</td>
<td>0.6689, 0.6062</td>
</tr>
<tr>
<td>|h(j\omega=60)</td>
<td>1.2190</td>
<td>1.2186</td>
<td>1.2625, 1.3217</td>
</tr>
<tr>
<td>Power</td>
<td>0.3604,</td>
<td>0.3634</td>
<td>0.3218, 0.491</td>
</tr>
<tr>
<td></td>
<td>0.4567</td>
<td>0.4536</td>
<td></td>
</tr>
</tbody>
</table>

Finally, to extract the worst case for the power measurement, which will result as a combination of the previous factors, it is a reasonable hypothesis to consider the resulting pdf to be Gaussian. Therefore, a confidence level of 99.7% can be obtained by selecting three times the standard uncertainty (notice that the assumption is confirmed by the results in Fig. 5), i.e.,

\[ u(x) = 3a. \]  

Substituting the input values of nominal current and voltage in (31) and (34), and solving for the power measurement, Table V can be obtained.

One of the issues in using the GUM’s combined uncertainty method is the choice of the appropriate coverage factor \( K \) for a desired confidence level. This is due to the fact that the shape of the resultant pdf is not known a priori and, in particular, may not be Gaussian [3], [13]. It is recommended, in such a case, that numerical methods, such as Monte Carlo, are considered [3], [12]. Analyzing the table, it is clear that the PCT method presented in [1] and detailed here does not suffer this drawback. To demonstrate this point, let us solve the same example considered throughout this paper for different levels of parametric uncertainty. As shown in Fig. 6, if the uncertainty on the primary measurements grows, the error of the combined standard uncertainty method with respect to the Monte Carlo method grows significantly, as expected. For the PCT approach, and vice versa, the error is always under 1%. This result could be further improved, assuming a higher number of terms for the PCT expansion.

![Fig. 6. Effects of increasing the level of uncertainty and the difference between the results obtained using Monte Carlo and combined uncertainty method (GUM) and Monte Carlo and PCT methods for the Sallen–Key filter example.](image)

![Fig. 7. (a) Open-circuit test to determine \( V_0 \) and (b) closed-circuit test to determine \( Z_a \).](image)

VII. IMPEDANCE MEASUREMENT

The first example allowed us to demonstrate the process; now, we focus on a more complex case with direct practical implications. The measurement of the loop impedance is a typical case of indirect measurement [2]. In [2], a calibration procedure based on an uninterruptible power supply (UPS) is proposed. The topology can be modeled with the circuit of Fig. 7, where \( E \) and \( Z_a \) represent the Thevenin equivalent of the closed-loop operations of the UPS.

The first measure is the open-loop voltage \( V_0 \). Then, a known shunt resistor is introduced, and the voltage across this resistor is measured. From this indirect measurement procedure, a governing equation can be expressed as

\[ Z_a = \frac{\tilde{V}_R R - \tilde{V}_0 R + \tilde{V}_R Z_i}{\tilde{V}_R}. \]  

Equation (36) can be rewritten as

\[
[\Re(V_R) + j\Im(V_R)] [\Re(Z_a) + j\Im(Z_a)]
+ [\Re(V_R) + j\Im(V_R)] R - V_0 R
+ [\Re(V_R) + j\Im(V_R)] [\Re(Z_i) + j\Im(Z_i)]
\]

where

- \( Z_a \) output impedance of the UPS;
- \( Z_i \) internal resistance of power source;
- \( R \) is a shunt resistor;
- \( V_R \) is the voltage measured across the shunt resistor, \( R \);
- \( V_0 \) is the open-loop voltage (see Fig. 7).

For this example, three cases can be considered, involving three parameters.

- Case 1: \( Z_i = 0 \), uncertainty in measurement due to the uncertainty of \( V_R \), and \( R \).
- Case 2: No uncertainty for \( V_R \), uncertainty due to \( Z_i \), and \( R \).
- Case 3: Uncertainty of \( V_R \), uncertainty due to \( Z_i \), and \( R \).
A general PCT expansion allows for the investigation of each case by considering each parameter as being uncertain. This method is not generally computationally efficient, but it exposes the generality of the PCT method. Therefore, from (37), the three uncertain parameters are $V_R$, $Z_t$, and $R$. However, $V_R$ and $Z_t$ are each expressed with two uncertain terms representing their real and imaginary parts. Thus, the PCT expansion and the Monte Carlo analysis are performed with five random variables, and the expansion of these variables with PCT results to

$$
\begin{align*}
\Re(V_R) &= \sum_{k=0}^{P} \Re(V_R)_k \Phi_k(\xi_1) \\
\Im(V_R) &= \sum_{k=0}^{P} \Im(V_R)_k \Phi_k(\xi_2) \\
\Re(Z_t) &= \sum_{k=0}^{P} \Re(Z_t)_k \Phi_k(\xi_3) \\
\Im(Z_t) &= \sum_{k=0}^{P} \Im(Z_t)_k \Phi_k(\xi_4) \\
R &= \sum_{k=0}^{P} R_k \Phi_k(\xi_5). \quad (38)
\end{align*}
$$

The variable of interest is $Z_a$, and it is dependent on all the uncertain variables; its expansion is given by

$$
\begin{align*}
\Re(Z_a) &= \sum_{k=0}^{P} \Re(Z_a)_k \Phi_k(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5) \\
\Im(Z_a) &= \sum_{k=0}^{P} \Im(Z_a)_k \Phi_k(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5). \quad (39)
\end{align*}
$$

Considering the LHS of (37) and substituting (38) and (39), it follows that (37) becomes

$$
\begin{align*}
\text{LHS} &= \left[ \sum_{a=0}^{P} \Re(V_R)_a \Phi_1(\xi_1) + j \sum_{b=0}^{P} \Im(V_R)_b \Phi_2(\xi_2) \right] \\
&\quad \times \left[ \sum_{c=0}^{P} \Re(Z_t)_c \Phi_c(\xi_1, \ldots, \xi_5) + j \sum_{d=0}^{P} \Im(Z_t)_d \Phi_d(\xi_1, \ldots, \xi_5) \right] \\
&\quad + \sum_{a=0}^{P} \Re(Z_a)_a \Phi_2(\xi_1, \ldots, \xi_5). \quad (40)
\end{align*}
$$

and the RHS can be expanded similarly as follows:

$$
\begin{align*}
\text{RHS} &= \left[ \sum_{a=0}^{P} \Re(V_R)_a \Phi_1(\xi_1) + j \sum_{b=0}^{P} \Im(V_R)_b \Phi_2(\xi_2) \right] \\
&\quad \times \left[ \sum_{c=0}^{P} \Re(Z_t)_c \Phi_c(\xi_1, \ldots, \xi_5) + j \sum_{d=0}^{P} \Im(Z_t)_d \Phi_d(\xi_1, \ldots, \xi_5) \right] \\
&\quad + \sum_{a=0}^{P} \Re(Z_a)_a \Phi_2(\xi_1, \ldots, \xi_5) + j \sum_{b=0}^{P} \Im(Z_a)_b \Phi_2(\xi_1, \ldots, \xi_5). \quad (41)
\end{align*}
$$

The PCT coefficients or both the LHS and RHS can be found by taking the following Galerkin projection:

$$
\begin{align*}
\text{LHS}_i &= \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} (LHS)(\Phi_i(\xi_1, \ldots, \xi_5))d\xi_1, \ldots, d\xi_5 \\
&= \frac{1}{10000} \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} (LHS)(\Phi_i(\xi_1, \ldots, \xi_5))^2 d\xi_1, \ldots, d\xi_5 \\
\text{RHS}_i &= \frac{1}{10000} \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} (RHS)(\Phi_i(\xi_1, \ldots, \xi_5))d\xi_1, \ldots, d\xi_5 \quad (42)
\end{align*}
$$

where $w$ is a weighting function determined by the choice of basis in the Askey scheme and the dimension of the variable $\Phi(\xi)$ (in this case 5).

The final step is the determination of the worst case by equating LHS = RHS, and solving for $Z_{ai}$, where $Z_{ai}$ is the representation of $Z_a$ in terms of PCT coefficients. In the following, we analyze three different cases for different hypotheses on the uncertain terms.

Case 1) $Z_t = 0$, uncertainty in measurement is due to the voltage measurement $V_R$, where $V_R$ is defined as an uncertain source with a uniform distribution within the interval [109.4 + j109.4, 109.52 + j109.52], and test resistance $R$ is uncertain with a uniform distribution within the interval [5.47, 5.476]. We have

$$
Z_a = (2.723 + j2.75) + (0.0 - j0.1507)\xi_1 \\
+ (0.001507 + j0.0)\xi_2 + (0.00149 + j0.001507)\xi_5. \quad (43)
$$

The pdf of the $|Z_a|$ obtained using the Monte Carlo and first-order PCT expansion methods can be seen in Fig. 8(a), where the Monte Carlo and PCT results match. The values at the tails of the pdf can be calculated by considering when the absolute value of (43) is maximum and minimum, as seen in Table VI.

Case 2) No uncertainty of the voltage measurement $V_R$ and $V_R = 109.46 + j109.46$, uncertainty due to $Z_t$, where $Z_t = [0.0029 + j0.0029, 0.0031 + j0.0031]$, and test resistance $R$, where $R = [5.47, 5.476]$. We have

$$
Z_a = (2.726 + j2.753) + (0.0 - j0.00010)\xi_4 \\
+ (0.00010 + j0.0)\xi_3 + (0.00149 + j0.0001507)\xi_5. \quad (44)
$$

The pdf of the $|Z_a|$ obtained using the Monte Carlo and first-order PCT expansion methods is shown in Fig. 8(b), where the Monte Carlo and PCT
Fig. 8. (a) Case 1: \( Z_i = 0 \), uncertainty due to accuracy of \( V_R \) and test resistance \( R \). (b) Case 2: no uncertainty on \( V_R \) and uncertainty due to \( Z_i \) and test resistance \( R \). (c) Case 3: uncertainty on \( V_R \) and uncertainty due to \( Z_i \) and test resistance \( R \).

### Table VI
Minimum, Expected, and Maximum of \( |Z_a| \) Calculated for Case 1: \( Z_i = 0 \), Uncertainty in Measurement is Due to the Accuracy of the Voltage Sensor \( V_R \) and Test Resistance \( R \)

<table>
<thead>
<tr>
<th></th>
<th>Expected-case</th>
<th>Maximum</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \xi_1 = -1, \xi_2 = -1 )</td>
<td>3.870</td>
<td>3.874</td>
</tr>
<tr>
<td>( \xi_1 = -1, \xi_2 = 0 )</td>
<td>3.866</td>
<td>3.874</td>
</tr>
<tr>
<td>( \xi_1 = -1, \xi_2 = 1 )</td>
<td>3.866</td>
<td>3.874</td>
</tr>
</tbody>
</table>

### Table VII
Minimum, Expected, and Maximum of \( |Z_a| \) Calculated for Case 2: No Uncertainty on the Voltage Sensor \( V_R \), Uncertainty Due to \( Z_i \) and Test Resistance \( R \)

<table>
<thead>
<tr>
<th></th>
<th>Expected-case</th>
<th>Maximum</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \xi_1 = -1, \xi_2 = -1 )</td>
<td>3.874</td>
<td>3.877</td>
</tr>
<tr>
<td>( \xi_1 = -1, \xi_2 = 0 )</td>
<td>3.872</td>
<td>3.877</td>
</tr>
<tr>
<td>( \xi_1 = -1, \xi_2 = 1 )</td>
<td>3.872</td>
<td>3.877</td>
</tr>
</tbody>
</table>

### Table VIII
Minimum, Expected, and Maximum of \( |Z_a| \) Calculated for Case 3: Uncertainty on \( V_R \), Uncertainty Due to \( Z_i \) and Test Resistance \( R \)

<table>
<thead>
<tr>
<th></th>
<th>Expected-case</th>
<th>Maximum</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \xi_1 = \xi_2 = \xi_3 = -1 )</td>
<td>3.874</td>
<td>3.879</td>
</tr>
<tr>
<td>( \xi_1 = \xi_2 = \xi_3 = 0 )</td>
<td>3.870</td>
<td>3.879</td>
</tr>
<tr>
<td>( \xi_1 = \xi_2 = \xi_3 = 1 )</td>
<td>3.870</td>
<td>3.879</td>
</tr>
</tbody>
</table>

### Table IX
Summary of the Results

<table>
<thead>
<tr>
<th>Parameter</th>
<th>PCT</th>
<th>Monte Carlo</th>
<th>Combined uncertainty (GUM)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case 1</td>
<td>3.866, 3.874</td>
<td>3.866, 3.874</td>
<td>3.866, 3.874</td>
</tr>
<tr>
<td>Case 2</td>
<td>3.872, 3.877</td>
<td>3.872, 3.877</td>
<td>3.869, 3.876</td>
</tr>
<tr>
<td>Case 3</td>
<td>3.870, 3.879</td>
<td>3.870, 3.879</td>
<td>3.868, 3.877</td>
</tr>
</tbody>
</table>

results match. The values at the tails of the pdf can be calculated through the maximum and minimum of the absolute value of (44). These results are shown in Table VII.

Case 3) Uncertainty of \( V_R \), where \( V_R \) is defined as an uncertainty source with a uniform distribution within the interval \([109.4 + j109.4, 109.52 + j109.52]\), uncertainty due to \( Z_i \), where \( Z_i \) is defined as an uncertain impedance with a uniform distribution within the interval \([0.0029 + j0.0029, 0.0031 + j0.0031]\), and uncertainty due to the test resistance \( R \), defined with a uniform distribution within the interval \([5.47, 5.476]\). We have

\[
Z_a = (2.726 + j2.753) + (0.0 - j0.001507)\xi_1 \\
+ (0.001507 - j2.059E - 14)\xi_2 \\
\times (0.00010 + j0.0)\xi_3 + (0.0 + j0.0001)\xi_4 \\
+ (0.00149 + j0.001507)\xi_5. \tag{45}
\]

The pdf of the \(|Z_a|\) obtained using the Monte Carlo and first-order PCT expansion methods are
shown in Fig. 8(c), where the Monte Carlo and PCT results match. The values at the tails of the pdf can be calculated through the maximum and minimum of the absolute value of (45). Results are shown in Table VIII.

In this example, the results for the three cases mentioned above can also be compared with the results obtained using the GUM. Table IX shows that the result from GUM matches reasonably well with that of Monte Carlo, but PCT shows increased accuracy.

This example demonstrates the generality of the PCT method, where all the key parameters can initially be expanded and the variation and/or the absence of specific uncertainties can be investigated.

VIII. Conclusion

This paper has outlined an analytical method based on PCT to estimate the propagation of uncertainty of an indirect measurement and the evaluation of the worst case. The worst-case uncertainty analysis focuses on the most likely case and the two extreme cases defining the tails of the pdf. The authors analyzed the identification of the worst-case condition in nonlinear uncertainty propagation cases. In particular, the exemplification of the method is shown in two applications. The first example is a power measurement in a 60-Hz system, where voltage and current measurements are conditioned with two second-order filters with uncertain parameters. The second example is an indirect impedance measurement. This paper has demonstrated that PCT can be a useful tool to quantify the uncertainty, and because PCT describes the pdf in the form of a polynomial function, the determination of maxima and minima for worst-case/best-case analysis can be performed relatively easily.

REFERENCES


Antonello Monti (S’88–M’89–SM’02) received the M.S. degree in electrical engineering and the Ph.D. degree from the Politecnico di Milano, Milan, Italy, in 1989 and 1994, respectively.

From 1990 to 1994, he was with the Research Laboratory, Ansoldo Industria, Milan, where he was responsible for the design of the digital control of a large power cycloconverter drive. In 1995, he joined the Department of Electrical Engineering, Politecnico di Milano, as an Assistant Professor. Since August 2000, he has been an Associate Professor with the Department of Electrical Engineering, University of South Carolina, Columbia. He is the author or a coauthor of more than 200 papers in the field of power electronics and electrical drives.

Prof. Monti served as the Chair of the IEEE Power Electronics Committee on Simulation, Modeling and Control and as an Associate Editor for the IEEE TRANSACTIONS ON AUTOMATION SCIENCE AND ENGINEERING.

Ferdinanda Ponci received the M.S. and Ph.D. degrees in electrical engineering from Politecnico di Milano, Milan, Italy, in 1998 and 2002, respectively.

In 2003, she joined the Department of Electrical Engineering, University of South Carolina, Columbia, as an Assistant Professor. She is currently with the Power and Energy Research Group, University of South Carolina. Her current research includes methods for uncertainty representation and propagation and multiagent systems for control and monitoring of power electronics systems.