AMUEM 2008 – International Workshop
on Advanced Methods for Uncertainty Estimation in Measurement
Sardagna, Trento, Italy, 21-22 July 2008

Confidence Interval Estimation using Polynomial Chaos Theory

A. Smith[1], A. Monti[1], F. Ponci[1]
[1]Department of Electrical Engineering, University of South Carolina
Columbia, SC (USA)
Phone: +1-803-777-2722, Fax: +1-803-777-8045, Email: [smith35, monti, ponci]@engr.sc.edu

Abstract **–** *This paper proposes an analytical method to estimate the confidence interval of a measurement using polynomial chaos theory (PCT). In previous work, the authors proposed an approach based on polynomial chaos theory (PCT) to evaluate the worst-case in an indirect measurement process, which in some cases can be considered as 100% confidence interval estimate. In many practical cases though, it is important to be able to determine the confidence intervals. The confidence interval computed as the integral the probability density function (PDF), the Cumulative Distribution Function (CDF), requires the availability of the PDF.*

An analytical approach to calculate of the PDF associated to a PCT polynomial has been previously proposed in literature. This analytical approach based on a regularized estimation of the joint PDF, yields a well-defined and well shaped PDF. The method proposed in this paper is based on the analytical reconstruction of PDF from the polynomial structure of the PCT expansion. Once the PDF is obtained, the integral can be calculated and an estimation of the confidence interval can be derived. This methodology is demonstrated using an indirect loop impedance measurement as an example.

Keywords *– uncertainty, electric variables measurement*

I. INTRODUCTION

In previous papers [1][2], the authors described using polynomial chaos theory (PCT) to evaluate the worst-case indirect measurement, which can be considered looking at the 100% confidence interval. But sometimes it is of great interest not to look at just the 100 % confidence level (as in the case of risk management or assessment).

In this paper the authors demonstrates the use of PCT to define different levels of confidence intervals from the PCT polynomial. PCT defines a spectral expansion of random variables that approximates the random process by means of a complete and orthogonal polynomial basis of random variables [3]. The result of this expansion is a polynomial that represents all the possible values the uncertain variable can assume as function of random variables with known distribution.

The approach taken in this paper is to use the polynomial described by the PCT expansion to generate an analytical expression for the PDF. From this analytical expression the cumulative distribution function (CDF) can then be calculated. Knowing analytically the CDF it is then possible to evaluate the confidence intervals.

The paper is structured as follows: first a brief introduction to PCT (section I) and the steps needed to expand a system using PCT (section II) are presented. Then, the description of the methodology used to calculate the PDF/CDF and confidence interval estimation from the PCT polynomial is presented. An example using an indirect impedance measurement is introduced in section V while section VI reports the conclusions.

II. POLYNOMIAL CHAOS THEORY

In 1938, N Wiener introduced, Homogeneous Chaos Expansion, discussing the use of Hermite polynomials and homogeneous chaos [4]. Wiener used the Hermite polynomials in a stochastic space to represent and propagate uncertainty [4]. This scheme was later expanded to include the whole Askey-scheme of orthogonal polynomials and was renamed Wiener-Askey Polynomial Chaos [3]. PCT is a spectral expansion of random variables that approximates the random process by a complete and orthogonal polynomial basis in terms of random variables with known PDF. The spectral expansion of a second order process using PCT can be described as:

$$X(\theta) = \sum_{i=0}^{\infty} a_i \Phi_i(\boldsymbol{\xi}(\theta)) \tag{1}$$

where: $X(\theta)$ is the uncertain variable under analysis in terms of θ

a_i are the coefficients of the expansion,

Φ_i are the polynomials of the selected base, and

ξ_{i_1} are random variables with a suitable PDF defined according to the polynomial base.

The spectral expansion is an infinite series and for practical purposes must be limited.

$$X(\theta) = \sum_{i=0}^{P} a_i \Phi_i(\boldsymbol{\xi}(\theta)) \tag{2}$$

Given the two values, the number of independent sources of uncertainty (n_v) and the maximum order for the polynomial base (n_p) the total number of terms, the value of P needed for the truncated PCT expansion is given as:

978-1-4244-2237-1/08/$25.00 ©2008 IEEE

$$P = \left(\frac{(n_v + n_p)!}{n_v! n_p!} \right) - 1 \tag{3}$$

PCT has found a place in many applications such as in the fields such as fluid dynamics [5], measurement uncertainty [1] [2][6][7][8], ,power electronics and circuit simulation [9], entropy multivariate analysis [10], control design [6][11]and polynomial chaos based observers for use in control theory [8]. In [16] an overview of applications of PCT to the electrical engineering domain was also discussed.

III. PCT EXPANSION

The PCT expansion of a system can be done in a few steps. These steps are as follows:

- Choosing the appropriate basis and PCT order
- Expanding the uncertainty variable(s)
- Substituting uncertainty variable(s) in the governing equation.
- Using the Galerkin projection to find the coefficients of the PCT expansion.

This process determines the creation of an expanded deterministic model that can be analyzed and executed by using traditional simulation methods. When the number of uncertainties grows significantly the Galerkin projection is usually substituted with the collocation approach. A comprehensive comparison between Galerkin approach and collocation method can be found in [17].

IV. PDF/CDF AND CONFIDENCE INTERVAL ESTIMATION

The development of the confidence estimator from the PCT polynomial consists of three steps:

- Formulate the PDF
- Formulate the CDF
- Use of the CDF to reconstruct the confidence interval

For the PDF formulation, consider the fact that the PCT expansion of a system is a function of random variables, $\boldsymbol{z}(\xi_1,..,\xi_n)$ where $\xi_1 \ldots \xi_n$ are independent random variables. This function $\boldsymbol{z}(\,)$ can be seen as a random variable with its PDF. Thus, it is possible to formulate the probability distribution of $\boldsymbol{z}$.

Given the PCT expansion of z:

$$\boldsymbol{z} = g(\xi_1,....,\xi_n) \tag{4}$$

its PDF, f_z, can be found by introducing the auxiliary variables. $w_x(\xi_1,..,\xi_n)$ These auxiliary variables are chosen such that the Jacobian of the transformation exists (see equation (5)).

$$J(\xi_1,...,\xi_n) = \begin{vmatrix} \frac{\partial g(\xi_1,...,\xi_n)}{\xi_1} & \cdots & \cdots & \frac{\partial g(\xi_1,...,\xi_n)}{\xi_n} \\ \frac{\partial w_1(\xi_1,...,\xi_n)}{\xi_1} & \cdots & \cdots & \frac{\partial w_1(\xi_1,...,\xi_n)}{\xi_1} \\ \vdots & \cdots & \cdots & \vdots \\ \frac{\partial w_{n-1}(\xi_1,...,\xi_n)}{\xi_1} & \cdots & \cdots & \frac{\partial w_{n-1}(\xi_1,...,\xi_n)}{\xi_n} \end{vmatrix} \tag{5}$$

After introducing the auxiliary variables, the joint distribution of z and w, must be found. The joint distribution of z and w_x, $\boldsymbol{f_{zw}(z,w)}$, involves finding the solution of the equation

$$g(\xi_1,...,\xi_n) = z \qquad h_x(\xi_1,...,\xi_n) = w_x \tag{6}$$

for $\xi_1 \ldots \xi_n$ in terms of z and w_x.

Let $\xi_{1x},\ldots, \xi_{nx}$ be solutions of equation (6) in terms of z and all w_x. If all of the solutions,($\xi_{1x},\ldots, \xi_{nx}$) are real functions then, [13]:

$$f_{zw}(z,w_1,\ldots,w_n) = \frac{f_{\xi_1,...,\xi_n}(\xi_{1_1},...,\xi_{n_1})}{|J(\xi_{1_1},...,\xi_{n_1})|} + \cdots + \frac{f_{\xi_1,...,\xi_n}(\xi_{1_n},...,\xi_{nn})}{|J(\xi_{1_n},...,\xi_{n_n})|} \tag{7}$$

where $f_{\xi_1,..\xi_n}$ is the joint probability density function of the independent random variables. If the solutions are not real, then [13]

$$f_{zw}(z,w_1,\ldots,w_n) = 0 \tag{8}$$

The PDF, f_z, of a PCT expansion, is then given as

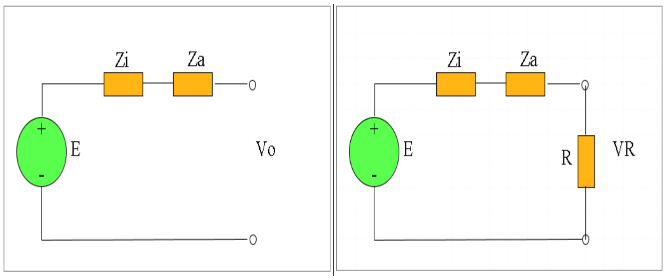
$$PDF(z) = f_z = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_{zw}(z,w_1,...,w_{n-1}) dw_1 \ldots dw_{n-1} \tag{9}$$

The CDF can then be found using:

$$CDF(z) = \int f_z dz \tag{10}$$

V. IMPEDANCE MEASUREMENT EXAMPLE

Consider the example presented in [1], where the circuit in Figure 1, is used to find the output impedance, Z_a. In this method, described in more details in [14], the open-loop circuit is used to measure V_0. Then, using a known shunt resistor, the voltage across this resistor was obtained.



(a) (b)

Figure 1:a) Open-circuit test to determine V_0. b) Closed-circuit test to determine Z_a

This indirect measurement process can be described by the following governing equation:

$$Z_a = \frac{\vec{V}_R R - \vec{V}_0 R + \vec{V}_R \vec{Z}_i}{\vec{V}_R} \tag{11}$$

Using the procedure described in section 3 and in [1], the PCT expansion for the Z_a can be obtained for both the real part and imaginary part of Z_a, given an uncertainty on the voltage measurement. For this example Z_i is considered to be real. The real and imaginary part of Z_a is given as

$$\begin{aligned} \Re(Z_a) &= 0.0275 + 1.556E-8\xi_2^2 - 1.556E-6\xi_1^2 \\ &\quad + 2.923\xi_1 + 5.191\xi_2 - 9.592\xi_1\xi_2 \\ \Im(Z_a) &= 0.545 - 4.796E-9\xi_2^2 + 4.796E-7\xi_1^2 \\ &\quad - 5.911\xi_1 + 2.926\xi_2 - 3.111E-7\xi_1\xi_2 \end{aligned} \tag{12}$$

Consider first the real part of Z_a, the PCT expansion. The equation (12) has two independent random variables which are uniformly distributed, ξ_1 and ξ_2 .

Introducing the auxiliary variable:

$$w_1 = \xi_1 \tag{13}$$

and solving the real part of equation (12) for ξ_2 in terms of w_1 and z as in equation (5), we find that the solutions , a_1 and a_2 ,for this example are:

$$\begin{aligned} a_{1,2} &= -1899.56 + 3.082 w_1 \\ &\pm 3.217E-10\sqrt{1.785E25 - 1.934E24 w_1 + 1.060E21 w_1^2 - 6.223E24 z} \end{aligned} \tag{14}$$

The density functions can be found by substituting the auxiliary variable and the solutions found in equation (14)

$$f_{\xi_1\xi_2}(w_1,a_1) = \left(\begin{cases} 0 & w_1 < -1 \\ \frac{1}{2} & w_1 < 1 \\ 0 & 1 \le w_1 \end{cases}\right)\left(\begin{cases} 0 & a_1 < -1 \\ \frac{1}{2} & a_1 < 1 \\ 0 & 1 \le a_1 \end{cases}\right)$$
$$f_{\xi_1\xi_2}(w_1,a_2) = \left(\begin{cases} 0 & w_1 < -1 \\ \frac{1}{2} & w_1 < 1 \\ 0 & 1 \le w_1 \end{cases}\right)\left(\begin{cases} 0 & a_2 < -1 \\ \frac{1}{2} & a_2 < 1 \\ 0 & 1 \le a_2 \end{cases}\right) \tag{15}$$

The Jacobian of the transformation is given as

$$J = -5.91E-5 - 3.112E-8\xi_2 + 9.592E-8\xi_1 \tag{16}$$

The PDF of the real part of Z_a can then be found using equations (7) (15) and (16) .A plot of the resulting PDF can be seen in Figure 2 and a comparison with Monte Carlo can be seen in Figure 3. The CDF of the real part of Z_a can then be found using equation (10) and a plot of the CDF can be seen in Figure 4.

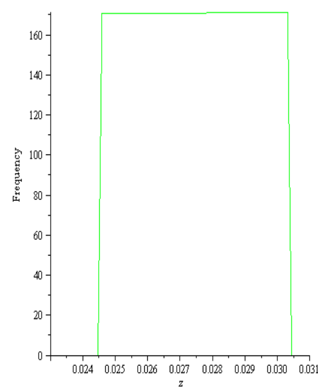


Figure 2: PDF obtained from the analytical expression of the PDF of the real part of the PCT expansion Za.

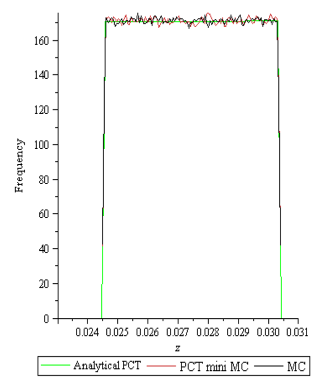


Figure 3: Comparison between the PCT PDF and the one obtained from Monte Carlo for the real part of Za.

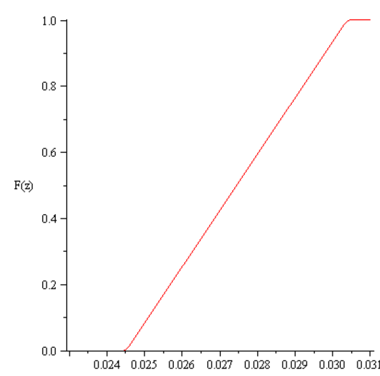


Figure 4: CDF of the real part Za

A similar procedure can be applied to the imaginary part where, by solving the imaginary part of equation (12) for ξ_2 in terms of w_1 and z, we find that the solutions, a_1 and a_2 are

$$a_{1,2} = 30504.189 - 32.441 w_1 \pm 8.341E-9\sqrt{1.50E25 - 3.022E22 w_1 + 1.657E19 w_1^2 - 3.0E24 z} \quad (17)$$

The density functions can be found as

$$f_{\xi_1\xi_2}(w_1,a_1) = \left(\begin{cases} 0 & w_1 < -1 \\ \frac{1}{2} & w_1 < 1 \\ 0 & 1 \le w_1 \end{cases}\right)\left(\begin{cases} 0 & a_1 < -1 \\ \frac{1}{2} & a_1 < 1 \\ 0 & 1 \le a_1 \end{cases}\right)$$
$$f_{\xi_1\xi_2}(w_1,a_2) = \left(\begin{cases} 0 & w_1 < -1 \\ \frac{1}{2} & w_1 < 1 \\ 0 & 1 \le w_1 \end{cases}\right)\left(\begin{cases} 0 & a_2 < -1 \\ \frac{1}{2} & a_2 < 1 \\ 0 & 1 \le a_2 \end{cases}\right) \quad (18)$$

The Jacobian of the transformation is given as

$$J = -2.925E-5 - 9.592E-8\xi_2 + 3.112E-8\xi_1 \quad (19)$$

And the plot of the resulting PDF can be seen in Figure 5 and a comparison with Monte Carlo can be seen in Figure 6. And the CDF can be seen in Figure 7.

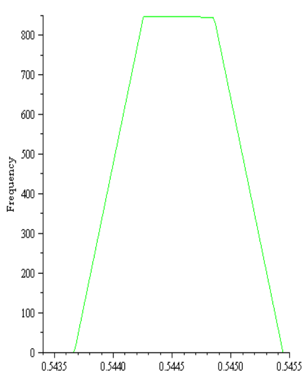


Figure 5: PDF obtained from the analytical expression of the PDF of the imaginary part of the PCT expansion Za

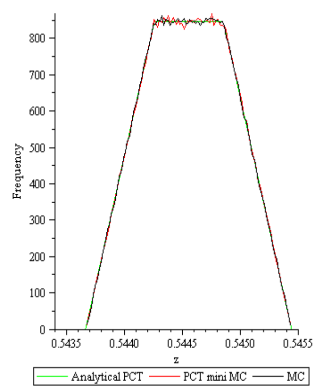


Figure 6: Comparison between the PCT PDF and the one obtained from Monte Carlo for the imaginary part of Za

Both a one-sided and two-sided confidence intervals can be estimated and can be obtained from the resulting CDF. For example, consider we need to find the confidence interval the indirect uncertainty impedance measurement i.e

$$P\{L \le \Re(Z_a) \le U\} = 1-\alpha$$
$$P\{L \le \Im(Z_a) \le U\} = 1-\alpha \quad (20)$$

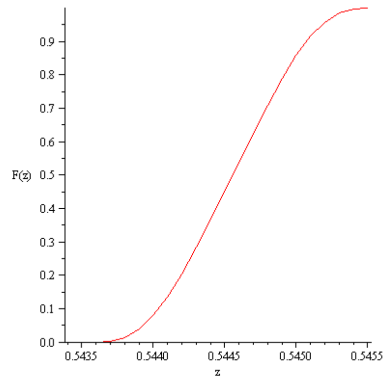


Figure 7: CDF of the imaginary part of Za

where equation (20) should be read as that the real and imaginary measurement of Z_a that “lies in the observed interval $[L,U]$ with confidence $100(1-\alpha)$”, [12] ($1-\alpha$ is called the confidence coefficient).

Let us now apply the previous results of the impedance measurement, to determine the confidence level in presence of uncertain voltage measurement, V_0. , In particular, the confidence level of the real part of Z_a within the interval [0.027, 0.029] and that of the imaginary part in the interval [0.5445, 0.5450] are determined. From Figure 8 and Figure 9 the confidence coefficient can be obtained from:

$$(1-\alpha) = F(z=U) - F(z=L) \quad (21)$$

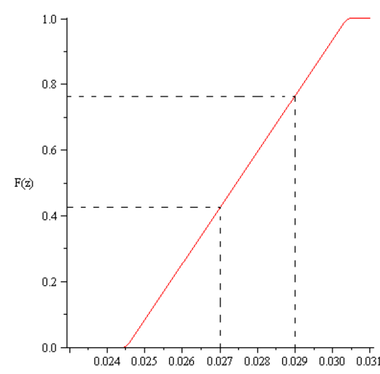


Figure 8: Using the CDF of the real part Za to obtain the confidence interval

The confidence of these measurement intervals are seen in Table 1 and Table 2.

Table 1: The confidence that the Re(Z_a) lies within the interval [0.027,0.029]

$[L,U]$	Confidence: 100(1-α)%
[0.027,0.029]	100(0.78-0.42)=36%

Table 2: The confidence that the Im(Z_a) lies within the interval [0.5445,0.5450]

$[L,U]$	Confidence: 100(1-α)
[0.5445,0.5450]	100(0.85-0.45)=40.0%

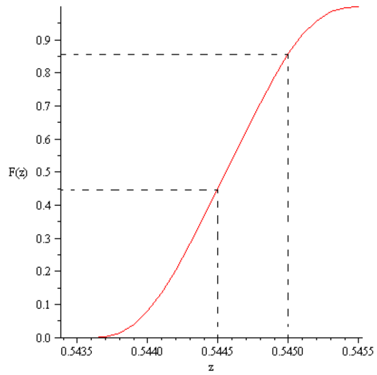


Figure 9 : Using the CDF of the imaginary part of Za to obtain the confidence interval

The above intervals, with specified lower and upper limit, are two-sided confidence intervals. However, sometimes it is appropriate to consider a lower confidence interval one-sided confidence interval:

$$\begin{aligned} &P\{L \le \Re(Z_a)\} = 1-\alpha \\ &P\{L \le \Im(Z_a)\} = 1-\alpha \end{aligned} \tag{22}$$

Or a upper confidence interval:

$$\begin{aligned} &P\{\Re(Z_a) \le U\} = 1-\alpha \\ &P\{\Im(Z_a) \le U\} = 1-\alpha \end{aligned} \tag{23}$$

The upper and lower (L,U) confidence limits can also be found for a given confident coefficient from the CDF.

As a final implementation note, it should be underlined that if too many independent sources of uncertainty are present, the proposed process may become too cumbersome. In particular, presently the analytical solution is obtained using Maple scripts, which may fail to find the roots of the solution as in equation (14) and (17). This is partly due to the limitations of Maple itself. We experienced that moving from version 10 to 11, some of the problems that appeared to be not solvable became actually feasible.

VI. CONCLUSION

In some applications the evaluation of the confidence interval in one single case does not provide enough information on the uncertainty of the measurement. It has been shown that it is possible to obtain a bounded worst-case value, which can be interpreted as a 100% confident interval.

The authors had demonstrated that PCT can be used to quantify the uncertainty of an indirect measurement and determine the worst-case. In this paper the authors demonstrate that from the PCT polynomial expansion, an analytical expression of the PDF can be found. It is then possible to analytically reconstruct the CDF and therefore determine the confidence level of a given interval or the interval corresponding to a given confidence level.

The proposed approach is demonstrated by means of a simple example of indirect measurement.

ACKNOWLEDGEMENT

This work was supported by the US Office of Naval Research under the Grant N00014-07-1-0603.

REFERENCE:

[1] A. Smith, A. Monti, F. Ponci, “Uncertainty and worst-case analysis for a low pass filter using Polynomial Chaos Theory”, IEEE AMUEM 2007, Trento Italy, July 2007.

[2] A. Smith, A. Monti, F. Ponci, “Uncertainty and worst-case analysis in electrical measurement using Polynomial Chaos Theory”, on printing for IEEE transaction on Instrument and Measurement,

[3] D. Xiu, & G.E. Karniadakis, The Wiener-Askey polynomial chaos for stochastic differential equations. SIAM J. Sci. Comput., vol. 24, pp. 619-644, 2002.

[4] N. Wiener, “The homogeneous chaos”, Amer J. Math, 1938, Vol. 60, pp. 897-936

[5] Xiu, D., Lucor, D., Su, C.-H. & Karniadakis, G.E. 2002. Stochastic modeling of flow-structure interactions using generalized polynomial chaos, J. Fluid Eng. Vol. 124: 51-59.

[6] T. Lovett, A. Monti, F. Ponci, “A Polynomial Chaos Theory Approach to the Control Design of a Power Converter”, 2004 IEEE Power Electronics Specialists Conference, pp. 4809-4813, June 20-25, 2004.

[7] T. Lovett, A. Monti, F. Ponci, “A Polynomial Chaos Approach to Measurement Uncertainty”, Instrumentation and Measurement, IEEE Transactions on Volume 55, Issue 3, June 2006 Page(s):729 – 736

[8] A. Smith, A. Monti and F. Ponci, "Indirect Measurements via a Polynomial Chaos Observer", IEEE Transactions on Instrumentation and Measurement, vol 56, no. 3, June 2007, 743-752.

[9] T. Lovett, A. Monti, F. Ponci, “A Polynomial Chaos Approach to Circuit Simulation Under Uncertainty”, on printing for Simulation Modelling Practice and Theory, Elsevier

[10] G. D’Antona, A. Monti, F. Ponci, L. Rocca, “Maximum Entropy Multivariate Analysis of Uncertain Dynamical Systems Based on the Wiener-Askey Polynomial Chaos”, IEEE transaction on Instrumentation and measurement, Vol. 56, No.3, June 2007.

[11] A Smith., A. Monti., F. Ponci, “Robust Controller Using Polynomial Chaos Theory”, on Proceeding of IEEE IAS 2006, Tampa, FL (USA), October 2006.

[12] W. Hines, D. Montgomery, D. Goldsman, C. Borror, “Probability and Statistics in Engineering, Fourth Edition”, John Wiley & Sons, Inc.,2003.

[13] A. Papoulis, “Probability, Random Variables and Stochastic Processes”, McGraw Hill, Inc, 1965.

[14] A Herrador and A Gonzalez, “Evaluation of measurement uncertainty in analytical assays by means of Monte-Carlo simulation,” Science direct, April 2004.

[15] D. Xiu, & G.E. Karniadakis, 2002b. Modeling uncertainty in flow simulations via generalized polynomial chaos. J. Comput. Phys. (to appear)

[16] A.Monti, F.Ponci, T.Lovett: "A Polynomial Chaos Theory Approach to Uncertainty in Electrical Engineering”, invited to IEEE Intelligent Systems Application to Power Systems (ISAP) 2005, Nov. 7-10, Washington DC, USA, pgg. 534-539.

[17] Prempraneerach, Pradya, “Uncertainty analysis in a shipboard integrated power system using multi-element polynomial chaos”, PhD Thesis, MIT, 2007 available at http://dspace.mit.edu/handle/1721.1/40364.