Uncertainty and Worst Case Analysis for a Low-Pass Filter Using Polynomial Chaos Theory

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Abstract — In this paper, the authors propose an analytical method to estimate the worst case in terms of uncertainty propagation in a measurement operation. The theory is exemplified with a specific application: power measurement with a signal conditioning stage including a low-pass filter. In AC power measurement the low pass filtering affects magnitude and phase of each harmonic component. The low pass filter is usually made of discrete components, which inherently contain parameter uncertainty. The effect of this uncertainty must be quantified to determine its effect on the quality of the power measurement. In this paper, the authors describe the use of Polynomial Chaos Theory (PCT) to quantify the effect of parameter uncertainty of a second order low pass filter such as the Sallen-Key filter. Furthermore, the authors demonstrate the use of PCT to obtain the probability density function (PDF) of the Sallen-Key filter given a certain parametric uncertainty. In particular it is shown how to use PCT to obtain the worst-case, expected and best-case output of the Sallen-Key filter without actually reconstructing the whole PDF. This result demonstrates the potential to determine significant boundary on the final measurement uncertainty.

Keywords — uncertainty, electric variables measurement

I. INTRODUCTION

The estimation of uncertainty is a critical step in any measurement process. Part of the measurement uncertainty can originate from parametric uncertainty in the electronics of the signal conditioning section. In particular pre-sampling conditioning is usually done by low-pass filtering for high-frequency noise reduction and/or as an anti-aliasing filter. Filtering operations are extremely important in power electronics devices in order to separate the switching noise from the main power flow.

The paper describes a method to quantify this contribution to overall measurement uncertainty using Polynomial Chaos Theory (PCT). It also provides a novel approach to use of PCT expansion to find the worst-case uncertainty determined by parameter uncertainty.

The authors introduce Polynomial Chaos Theory and then focus on the analysis of the theoretical foundation of using PCT expansion to estimate the worst-case.

In general, the main interest of the uncertainty analysis concentrates on the following parameters:

- The most likely case
- The two extreme cases defining the tail of the PDF

The tails of the PDF in many cases coincide with the worst cases. It should be pointed out that sometimes one of the tails of the PDF can be actually considered as the best case out of the total distribution. For example if the filter under analysis is assumed to be part of a closed-loop control system one of the tails gives the minimum phase shift in the frequency domain (best case) while the other gives the maximum phase shift (worst case).

Purely from a measurement perspective, both cases can be considered worst-case in the sense that represent the maximum distance from the most likely value.

The paper is structured as follows: in Section II a brief introduction of PCT is presented followed in Section III by the theory of worst-case analysis in the PCT domain. Section IV provides the uncertainty analysis of a second order Sallen-Key filter, while in Section V the propagation of the uncertainty in Sallen-Key on a power measurement is quantified. Also, a demonstration is presented on how to use PCT to obtain the worst-case measurement without reconstructing the PDF. Final conclusions are reported in Section VI.

II. POLYNOMIAL CHAOS THEORY

In 1938, N Wiener introduced, Homogeneous Chaos Expansion, where he discussed the use of Hermite polynomials and homogeneous chaos [1]. Wiener used the Hermite polynomials in a stochastic space to represent and propagate uncertainty in the form of a probability distribution function (PDF). This scheme was later expanded to include the whole Askey-scheme of orthogonal polynomials and was renamed Wiener-Askey Polynomial Chaos [2]. PCT is a spectral expansion of random variables that approximates the random process by a complete and orthogonal polynomial basis in terms of certain random variables. The spectral expansion of a second order random process using PCT can be described as:

\[ X(\theta) = \sum_{i=0}^{\infty} a_i \Phi_i(\xi(\theta)) \]

where: \( X(\theta) \) is the random process or function under analysis in terms of \( \theta \)

- \( a_i \) are the coefficients of the expansion,
- \( \Phi_i \) are the polynomials of the selected base, and
- \( \xi_i \) are random variables with a suitable PDF
defined according to the polynomial base.
The spectral expansion is an infinite series and for practical purposes must be limited.

\[ X(\theta) = \sum_{i=0}^{P} a_i \Phi_i(\xi(\theta)) \]  

(2)

Given the two values, the number of independent sources of uncertainty \((n_v)\) and the maximum order for the polynomial base \((n_p)\) the total number of terms, the value of \(P\) needed for the truncated PCT expansion is given as:

\[ P = \frac{(n_v + n_p)!}{n_v!n_p!} - 1 \]

(3)

Polynomial Chaos has been applied to numerous fields of study, including fluid dynamics and circuit simulation [3][5][6]. A previous application involving measurement is reported in [7].

III. WORST-CASE ANALYSIS

The use of PCT to represent a random variable as polynomial series has already been demonstrated [1][2][3][4][6][7]. One of the key aspects of this paper is the estimation of the worst-case, i.e. the analysis of the polynomial function in order to estimate the tails of a PDF. A generic random variable function of an uncertain parameter can be expressed through a PCT expansion such as

\[ f(\xi) = \sum_{n=0}^{P} x_n \Phi_n(\xi) \]

(4)

where \(f(\xi)\) is the set of all possible values assumed by the variable \(f\) as function of the random variable \(\xi\)

\(x_n\) are the coefficients of the PCT expansion

\(\Phi_n(\xi)\) is one of the orthogonal polynomials of the basis used in the PCT expansion

\(P\) is the number of terms in PCT expansion

**General Case**

From equation 4 the extreme cases can be expressed as:

\[ \sup_{\Omega} f(\xi) = \sup_{\Omega} \left( \sum_{n=0}^{P} x_n \Phi_n(\xi) \right) \]

(5)

\[ \inf_{\Omega} f(\xi) = \inf_{\Omega} \left( \sum_{n=0}^{P} x_n \Phi_n(\xi) \right) \]

where \(\Omega\) is the region where \(\Phi_n(\xi)\) is defined.

The calculation in equation (5) requires finding the value(s) of \(\xi\) that makes the expanded variable under consideration maximum or minimum.

In many cases this problem can be solved analytically, given the polynomial nature of the base functions and the complexity of the calculation is mostly related to the number of terms used in the PCT expansion. In particular, given the monotonic characteristics of many base functions, the problem becomes an easy constrained problem on the frontier of the domain of the variable \(\xi\).

In the following, we will focus on an application example to demonstrate the concept. Most of the computational effort required in this case has been automatized by means of Maple scripts, opening the door for easy extension to other cases.

IV. SALLEN-KEY FILTER

Let us consider the Sallen-Key filter with topology shown in Figure 1 and with the following transfer function:

\[ H(s) = \frac{1}{1 + C_2(R_1 + R_2)s + C_1C_2R_1R_2s^2} \]

(6)

The magnitude of the frequency response of this function is:

\[ |H(j\omega)| = \left| \frac{1 - C_1C_2R_1R_2\omega^2}{F(\omega)} \right| + \left| \frac{-C_2R_1\omega - C_2R_2\omega}{F(\omega)} \right|^2 \]

(7)

where

\[ F(\omega) = 1 + (-2C_2C_1R_1R_2 + C_2^2R_2^2 + 2C_2C_1R_1R_2 + C_1^2R_2^2)\omega^2 + C_1^2C_2^2R_1R_2^2\omega^4 \]

(8)

The phase of the frequency response is:

\[ \angle H(j\omega) = -\tan^{-1}\left( \frac{-C_2R_1\omega - C_2R_2\omega}{1 - C_1C_2R_1R_2\omega^2} \right) \]

(9)

We can split the analysis in the PCT domain in two parts

1. Impact of the parameter uncertainty on the amplitude
2. Impact of the parameter uncertainty on the phase

Where the uncertain parameter may be any of the lumped equivalent circuit parameters. Let us first analyze the phase. Consider the function \(f(\omega)\), such that \(\angle H(j\omega) = -\tan^{-1}(f(\omega))\):

\[ f(\omega) = \frac{-C_2R_1\omega - C_2R_2\omega}{1 - C_1C_2R_1R_2\omega^2} \]

(10)

In order to properly manage the division operation in the PCT domain, it is convenient to rewrite equation (9) as follows
Each side of the equation can be expanded separately and later equated and solved for \( f(\omega) \).

\[
\text{LHS} = (1 - C_1C_2R_2\omega^2) f(\omega) - C_2R_2\omega - C_3R_2\omega = 0
\]

(11)

\[
\text{RHS} = -C_2R_1\omega - C_3R_2\omega
\]

(12)

The next step is the substitution of the uncertain variables with their polynomial expression in the PCT domain. In this case, we assume that the uncertain variables are \( C_1, C_2, R_1 \) and that they are uniformly distributed. Since we assume the PDFs of the uncertain parameters to be uniform, then, according to [2] the basis of the PCT expansion is the Legendre basis.

The expansion of these uncertain variables can be expressed as:

\[
C_1 = \sum_{i=0}^{P} C_{1i}\xi_i
\]

\[
C_2 = \sum_{i=0}^{P} C_{2i}\xi_i
\]

(13)

\[
R_1 = \sum_{i=0}^{P} R_{1i}\xi_i
\]

Due to the uncertainty of the parameters, function \( f(\omega) \) is also uncertain. Let us also suppose to perform the analysis for a specific frequency, i.e. 60 Hz.

The PCT expansion for both the LHS and RHS can be found by taking the Galerkin projection:

\[
\text{LHS}_i = \frac{\int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} (\text{LHS})(w) \Phi_i(\xi_1, \xi_2, \xi_3) \, dw}{\int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} \Phi_i(\xi_1, \xi_2, \xi_3) \, dw}
\]

\[
\text{RHS}_i = \frac{\int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} (\text{RHS})(w) \Phi_i(\xi_1, \xi_2, \xi_3) \, dw}{\int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} \Phi_i(\xi_1, \xi_2, \xi_3) \, dw}
\]

(15)

where \( w \) is a weighting function determined by the choice of basis in the Askey scheme and the dimension of the variable \( \xi \) in this case three.

The final step is the determination of the worst case: it requires equating \( \text{LHS}_i = \text{RHS}_i \) and solving for \( f(\omega) \).

Let us now assume the following characteristics of the uncertain parameters:

- capacitor \( C_1 \): uniform distribution centered in 20\( \mu \)F, spanning 0.5\( \mu \)F
- capacitor \( C_2 \): same values as \( C_1 \)
- resistor \( R_1 \): uniform distribution centered in 100\( \Omega \) and spanning 10\( \Omega \)
- resistor \( R_2 \): 100\( \Omega \)

Table 1: PCT coefficients obtained from the PCT expansion of \( f(\omega=2\pi60) \)

<table>
<thead>
<tr>
<th>PCT Order</th>
<th>Coefficients</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st</td>
<td>([-3.526,-0.6396,-0.2042,-0.1192,-0.0566])</td>
</tr>
<tr>
<td>2nd</td>
<td>([-3.527,-0.6439,-0.2073,-0.1192,-0.0566,\ldots,-0.04864,-0.0138])</td>
</tr>
</tbody>
</table>

The coefficients of the PCT expansion of the uncertain parameters, computed as in [2] are reported in Table 1. Thus, \( f(\omega) \), reconstructed from \( f(\omega_i) \), is given as:

\[
f(\omega = 2\pi60, \xi) = -3.496 - 0.6439\xi_3 - 0.2073\xi_2 - 0.1192\xi_1
\]

\[
-0.08464\xi_2^2 - 0.006903\xi_2^2 - 0.003984\xi_1^2 - 0.0646\xi_2\xi_3
\]

\[
-0.04864\xi_1\xi_3 - 0.0138\xi_1^2\xi_2
\]

(17)

It can be shown that the max and min points of \( f \) occurs at the boundaries, therefore the extreme cases occur when \( \xi_1 = \xi_2 = \xi_3 = 1 \), \( \xi_1 = \xi_2 = \xi_3 = -1 \). Table 2 shows the values of \( f(\omega) \) at these boundary conditions.

Table 2: The best-case, expected and worst-case calculated from the PCT expansion of \( f(\omega=2\pi60) \)

<table>
<thead>
<tr>
<th>Case</th>
<th>Best-case</th>
<th>Expected-case</th>
<th>Worst-case</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \xi_1 = \xi_2 = \xi_3 = 1 )</td>
<td>-2.757</td>
<td>-3.496</td>
<td>-4.689</td>
</tr>
</tbody>
</table>

To obtain the PCT expansion of \( H(j\omega) \) the angle function

\[
\angle H(j\omega) = -\tan^{-1}(f(\omega))
\]

(18)

is expanded using the Taylor expansion thus resulting in:

\[
\angle H(j\omega) = -\tan^{-1}(f(\omega_0)) + \frac{f(\omega_0) - f(\omega_0)}{1 + f(\omega_0)^2} + \ldots
\]

(19)

where \( f(\omega_0) \) is a constant and is the expected value of the function not affected by uncertainty.

The uncertain variable in equation (19) is \( f(\omega) \) and its PCT expansion is given in equation (17).
(17) into equation (19), we can obtain the system equation for the PCT expansion. This system equation is given as

\[ \angle H(j\omega) = \frac{\int_{-1}^{1} \int_{-1}^{1} \left( \Phi \right) (w) d\Phi_n}{\int_{-1}^{1} \int_{-1}^{1} (\Phi_n)(\Phi) (w)} \]  

(20)

The distribution of \( \angle H(j\omega) \) for the same set of parameter as in what above can be seen in Figure 3.

The second order, three variable, PCT expansion coefficients are given in Table 3.

<table>
<thead>
<tr>
<th>PCT Order</th>
<th>Coefficients</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st</td>
<td>[0.6412, -0.0228, -0.1042, 0.01234]</td>
</tr>
<tr>
<td>2nd</td>
<td>[0.6413, -0.0227, -0.1053, 0.0131, 0.0000776, 0.008214, -0.000084, -0.0008356, -0.001708, -0.006978]</td>
</tr>
</tbody>
</table>

The PCT expansion of the phase of the frequency response is here reported for the polynomials up to the second order:

\[
\tan^{-1}(f(\omega = 2\pi 60, \xi)) = 1.292 + 0.04808 \xi_3 + 0.01532 \xi_2^2 + 0.008698 \xi_1 - 0.001104 \xi_2 - 0.0001957 \xi_3 + 0.00002427 \xi_1 \xi_3 - 0.00002703 \xi_1 \xi_2 + 0.0007409 \xi_1^2 - 0.00003605 \xi_2^2 - 0.0002703 \xi_2 \xi_3 + 0.0007409 \xi_1 \xi_3 + 0.0001182 \xi_1 \xi_2^2
\]

(21)

It can be shown that the max and min points occurs at the boundary condition, therefore the extreme cases occur when \( \xi_1 = \xi_2 = \xi_3 = 1 \), \( \xi_1 = \xi_2 = \xi_3 = -1 \). Table 4 shows the extreme cases for \( \angle H(j\omega = 2\pi 60) \).

<table>
<thead>
<tr>
<th>Best-case</th>
<th>Expected-case</th>
<th>Worst-case</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \xi_1 = \xi_2 = \xi_3 = 1 )</td>
<td>( \xi_1 = \xi_2 = \xi_3 = 0 )</td>
<td>( \xi_1 = \xi_2 = \xi_3 = -1 )</td>
</tr>
<tr>
<td>1.219</td>
<td>1.292</td>
<td>1.363</td>
</tr>
</tbody>
</table>

Now let us consider the function describing the magnitude as reported in equation (7). This equation can be re-written as

\[
|H(j\omega)|F(\omega) = \sqrt{\left(1 - C_1 \xi \omega - C_2 \xi^2 \omega^2 \right)^2 + \left(C_2 \xi \omega - C_2 \xi^2 \omega^2 \right)^2}
\]

(22)

Following the procedure described herein in this paragraph, the PCT expansion of \( H \) can be computed. Then, solving for \( |H(j\omega)|_n \), the PCT expansion for the magnitude of the frequency response can be obtained. Table 5 shows the coefficient of the PCT expansion for the magnitude.

<table>
<thead>
<tr>
<th>PCT Order</th>
<th>Coefficients</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st</td>
<td>[0.6412, -0.0228, -0.1042, 0.01234]</td>
</tr>
<tr>
<td>2nd</td>
<td>[0.6413, -0.0227, -0.1053, 0.0131, 0.0000776, 0.008214, -0.000084, -0.0008356, -0.001708, -0.006978]</td>
</tr>
</tbody>
</table>

\[
|H(j(\omega = 2\pi 60), \xi)| = 0.6376 - 0.02273 \xi_3 - 0.1053 \xi_2 + 0.01313 \xi_1 + 0.0001164 \xi_1 \xi_3 - 0.01232 \xi_2^2 - 0.001261 \xi_1 \xi_2 - 0.0008356 \xi_2 \xi_3 + 0.00170830 \xi_1 \xi_3 - 0.006978 \xi_1 \xi_2
\]

(23)

It can be shown that the max and min points do not occur when \( \xi_1 = -1, \xi_2 = 1, \xi_3 = 1 \), \( \xi_1 = \xi_2 = -1, \xi_3 = -1 \). Table 6 shows the extreme cases for \( |H(j(\omega = 2\pi 60)|_n \).
Table 6: The best-case, expected and worst-case calculated from the PCT expansion of $|H(j\omega = 2\pi 60)|$

<table>
<thead>
<tr>
<th></th>
<th>Best-case</th>
<th>Expected-case</th>
<th>Worst-case</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\xi_1 = 1, \xi_2 = \xi_3 = -1$</td>
<td>$0.6758$</td>
<td>$0.6376$</td>
<td>$0.6003$</td>
</tr>
</tbody>
</table>

The best-case power measurement for this example is given as

$$P_{\text{best-case}} = \left| V_{\text{best-case}} \right| \left| I_{\text{best-case}} \right| \cos(\theta_{\text{best-case}}) = (1)(0.6758) \times (1)(0.6758) \cos(0) = 0.4567W$$

From the previous results we have that the power measurement would be subject to an uncertainty that is roughly 100% of the expected value.

This uncertainty contribution can be reduced by calibrating the gain of the Sallen-Key filter. Therefore, the parametric uncertainty only affects the phase between the current and voltage. The worst-case power measurement thus becomes:

$$P_{\text{worst-case}} = \left| V \right| \left| I \right| \cos(\theta_{\text{worst-case}}) = (1)(0.6376) \times (1)(0.6376) \cos(1.219) = 0.999997W$$

And the worst-case percent error is 0.0033%.

VI. CONCLUSION

Quantifying the uncertainty introduced by the measurement chain components such as a low-pass filter is an important assessment for the validity of the measurement. This is especially important in the power measurement where there may be two independent working low-pass filters. This paper demonstrated that PCT can be a useful tool to quantify this uncertainty and because PCT describes the PDF in the form of a polynomial, analyzing the maximum and minimum for this polynomial worst-case, best-case analysis can be performed.

VII. ACKNOWLEDGMENT

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